

2 x 2 Matrix Gymnastics^{Br1}

The general form of the transport matrix written in terms of the Twiss parameters and the betatron phase advance is,

$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = \begin{pmatrix} \frac{\cos \Delta\psi + \alpha_1 \sin \Delta\psi}{\sqrt{\beta_1 / \beta_2}} & \frac{\sqrt{\beta_1 \beta_2} \sin \Delta\psi}{\sqrt{\beta_1 / \beta_2}} \\ \frac{(1 + \alpha_1 \alpha_2) \sin \Delta\psi + (\alpha_2 - \alpha_1) \cos \Delta\psi}{-\sqrt{\beta_1 \beta_2}} & \frac{\cos \Delta\psi - \alpha_2 \sin \Delta\psi}{\sqrt{\beta_2 / \beta_1}} \end{pmatrix} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

Writing the 2 x 2 matrix to transform (x, x') as,

$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

the Twiss parameters transform as follows,

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} R_{11}^2 & -2R_{11}R_{12} & R_{12}^2 \\ -R_{11}R_{21} & 1 + 2R_{12}R_{21} & -R_{12}R_{22} \\ R_{21}^2 & -2R_{21}R_{22} & R_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

Using the matrix for one full turn of a ring the betatron tune and Twiss parameters can be obtained as follows,

$$\nu = \frac{1}{2\pi} \cos^{-1} \left[\frac{\text{Tr} R}{2} \right] \quad \beta = \frac{R_{12}}{\sin(2\pi\nu)}$$

$$\alpha = \frac{R_{11} - R_{22}}{2 \sin(2\pi\nu)} \quad \gamma = \frac{-R_{21}}{\sin(2\pi\nu)}$$